Momentum Space
Coulomb Distorted Matrix Elements
for Heavy Nuclei

Ch. Elster

N. Upadhyay, V. Eremenko, F. Nunes, L. Hlophe,
G. Arbanas, J. E. Escher, I.J. Thompson

(The TORUS Collaboration)
What Reactions are we interested in?

Reactions: Elastic Scattering, Breakup, Transfer

$^3\text{He}(d,p)^4\text{He}$

$^{140}\text{Sn}(d,p)^{141}\text{Sn}$

Light nuclei

Heavy Nuclei

Degrees of Freedom?
Reduce Many-Body to Few-Body Problem

Task:
- Isolate important degrees of freedom in a reaction
- Keep track of important channels
- Connect back to the many-body problem

Hamiltonian for effective few-body problem:
\[ H = H_0 + V_{np} + V_{nA} + V_{pA} \]

- np interaction
- Optical potentials p+A and n+A

Three-Body Problem
(d,p) Reactions as three-body problem

Elastic, breakup, rearrangement channels are included and fully coupled (compared to e.g. CDCC calculations)

Issue: current momentum space implementation of Coulomb interaction (shielding) does not converge for $Z \geq 20$

Courtesy: F.M. Nunes
Solve Faddeev equations in Coulomb basis (no screening)

Implies integrals like:

\[
Z_l^C(p, p_\alpha) = \int \frac{dp'p'^2}{2\pi} U_i(p, p') \psi_i^C, p_\alpha(p')
\]

If

\[
U_i(p, p') = \sum_{i,j} u_{i,i}'(p) (M_{i,j})_{i,j} u_{i,j}(p')
\]

Integral contains smooth function \( u_{i,i}'(p') \) and \( \psi_{i\alpha}^C(p') \)

Coulomb wave function in momentum space and pw decomposition

Very nasty! “pole” at \( p_\alpha = p' \)

Suggestion is new & needs to be tested
First Test in Two-Body System

Calculate two-body Coulomb distorted nuclear matrix element

Separable nuclear Optical Potential

\[ u_l(p'_\alpha, p_\alpha) = \sum_{ij} u^*_l(p'_\alpha) [M_i]_{ij} u_j(p_\alpha) \]

\[ u_l(p_\alpha) \] is the nuclear potential form factor.

**Compute:** Coulomb distorted nuclear form factor

\[ u^C_l(p_\alpha) = \frac{1}{2\pi^2} \int dp p^2 u_l(p) \psi^C_{p\alpha l}(p) \]

\[ \psi^C_{p\alpha l}(p) \] is the Coulomb scattering wave function.
Challenges:

\[
\psi_{p\alpha l}^C(p) = -\frac{4\pi}{p} e^{-\pi \eta / 2} \Gamma(1 + i\eta) e^{i\alpha l} \left[ \frac{(p + p\alpha)^2}{4pp\alpha} \right]^l \\
\times \text{Im} \left[ e^{-i\alpha l} \frac{(p + p\alpha + i0)^{-1+i\eta}}{(p - p\alpha + i0)^{1+i\eta}} {}_2F_1 \left(-l, -l - i\eta; 1 - i\eta; \frac{(p - p\alpha)^2}{(p + p\alpha)^2}\right) \right]
\]

\[\eta = Z_1 Z_2 e^2 \mu / p\alpha.\]

- Compute special functions of complex arguments
- \( {}_2F_1(a,b;c,z) \) requires two different representations for pole and non-pole regions
- “oscillatory” singularity at \( p = p\alpha \)
- Gel’fand-Shilov regularization
  - Reduce integrand around pole by subtracting 2 terms of the Taylor series
Gel’fand-Shilov Regularization:
Generalization of Principal value regularization
Idea: reduce value of integrand near singularity

\[
\int_{-\Delta}^{\Delta} dx \frac{\varphi(x)}{x^{1+i\eta}} = \int_{-\Delta}^{\Delta} dx \frac{\varphi(x) - \varphi(0) - \varphi'(0)x}{x^{1+i\eta}}
- \frac{i\varphi(0)}{\eta} [\Delta^{-i\eta} - (\Delta)^{-i\eta}] + \ldots
\]

With Yamaguchi-type test form factor

First calculation of Coulomb distorted $^{208}\text{Pb}$ formfactor in momentum space!
\[ u_l^C(p_\alpha) = \int_{0}^{p_\alpha-\Delta} \frac{dp}{2\pi^2} p^2 u_l(p) \psi_{p_\alpha,l}^C(p) + \int_{p_\alpha-\Delta}^{p_\alpha+\Delta} \ldots + \int_{p_\alpha+\Delta}^{\infty} \ldots \]

**Fixed** \( p_\alpha \)

**Pole region**
$\mathbf{p} + ^{208}\text{Pb}$

\[ u_l^C(p_\alpha) = \int_0^{p_\alpha-\Delta} \frac{dp}{2\pi^2} p^2 u_l(p) \psi_{p_\alpha,l}^C(p) + \int_{p_\alpha+\Delta}^{p_\alpha+\Delta} \cdots + \int_{p_\alpha+\Delta}^{\infty} \cdots \]

Fixed $p_\alpha$
Reduce Many-Body to Few-Body Problem

Task:
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Hamiltonian for effective few-body problem:
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Optical potentials p+A and n+A

Three-Body Problem
Separable Representation of Optical Potentials

Starting point: Woods-Saxon Representation
Method of Ernst-Shakin-Thaler

**BUT:** *Needs to be generalized for complex potentials*

So that

\[ KUK^{-1} = U^\dagger \]

\[ K \] is the time-reversal operator.

\[ U = \sum_{i,j} u |\Psi_i^{(+)}\rangle \langle \Psi_j^{(+)}| M |\Psi_j^{(-)}\rangle \langle \Psi_j^{(-)}| u \]

\[ \delta_{ik} = \sum_j \langle \Psi_i^{(+)}| M |\Psi_j^{(-)}\rangle \langle \Psi_j^{(-)}| u |\Psi_k^{(+)}\rangle = \sum_j \langle \Psi_i^{(-)}| u |\Psi_j^{(+)}\rangle \langle \Psi_j^{(+)}| M |\Psi_k^{(-)}\rangle. \]

Definition with In/Out-states necessary to fulfill reciprocity theorem

**t-matrix:**

\[ t(E) = \sum_{i,j} u |\Psi_i^{(+)}\rangle \tau_{ij}(E) \langle \Psi_j^{(-)}| u \]

\[ \sum_j \tau_{ij}(E) \langle \Psi_j^{(-)}| u - u g_0(E) u |\Psi_i^{(+)}\rangle = \delta_{ik}. \]

*Compute and solve system of linear equations*
$n + ^{48}\text{Ca}$ and $n + ^{208}\text{Pb}$ : $l=0$

Chapel-Hill Optical Potential
Summary & Outlook

Faddeev-AGS framework in Coulomb basis passed first test!

- Momentum space nuclear form factors obtained in a Coulomb distorted basis for high charges for the first time.
- “Oscillatory singularity” of $\psi_{p_{\alpha}l}^c(p)$ at $p = p_{\alpha}$ successfully regularized.
- Algorithms to compute $\psi_{p_{\alpha}l}^c(p)$ and the overlap integral successfully implemented.

In Progress:

Calculations with separable p+A optical potentials (generalized EST scheme)

Near Future:

Implementation of Faddeev-AGS equations in the Coulomb basis to obtain (d,p) observables.
TORUS: Theory of Reactions for Unstable Isotopes

A Topical Collaboration for Nuclear Theory

http://www.reactiontheory.org/

- Ch. Elster, V. Eremenko†‡, and L. Hlophe†: Institute of Nuclear and Particle Physics, and Department of Physics and Astronomy, Ohio University, Athens, OH 45701.

- F. M. Nunes and N. J. Upadhyay†: National Superconducting Cyclotron Laboratory and Department of Physics and Astronomy, Michigan State University, East Lansing, MI 48824.

- G. Arbanas: Nuclear Science and Technology Division, Oak Ridge National Laboratory, Oak Ridge, TN 37831.


† Post-Docs or Grad Students.
‡ M. V. Lomonosov Moscow State University, Moscow, 119991, Russia.
Some insights for momentum space Coulomb wave functions:

**Pole:**

\[
\psi^C_{p\alpha l}(p) = -\left(\frac{4\pi}{p}\right) e^{-\pi\eta/2} \Gamma(1 + i\eta) e^{i\alpha_l} \left[\frac{(p + p_\alpha)^2}{4pp_\alpha}\right]^l \times \text{Im} \left[ e^{-i\alpha_l} \frac{(p + p_\alpha + i0)^{-1+i\eta}}{(p - p_\alpha + i0)^{1+i\eta}} \binom{2F1}{-l, -l - i\eta; 1 - i\eta; \zeta \equiv \frac{(p - p_\alpha)^2}{(p + p_\alpha)^2}} \right]
\]

Switching point: \( \zeta = \chi \approx 0.34 \)

\( \eta = Z_1 Z_2 e^2 \mu / p_\alpha \)

**Non-Pole:**

\[
\psi^C_{p\alpha l}(p) = -\left(\frac{4\pi\eta e^{-\pi\eta/2} p_\alpha (pp_\alpha)^2}{(p^2 + p_\alpha^2)^{l+1+i\eta}}\right) \left[\frac{\Gamma(l + 1 + i\eta) \Gamma(1/2)}{\Gamma(l + 3/2)}\right] \times \left[\left[p^2 - (p_\alpha + i0)^2\right]^{-1+i\eta}\binom{2F1}{\frac{l + 2 + i\eta}{2}, \frac{l + 1 + i\eta}{2}; l + \frac{3}{2}} \chi \equiv \frac{4p^2 p_\alpha^2}{(p^2 + p_\alpha^2)^2}\right]
\]
Some insights on momentum space Coulomb wave functions:
There are two representations for pole and non-pole regions

\[
\psi_{p_\alpha l}^C(p') = \frac{-4\pi e^{-\eta_\alpha \pi/2}}{p'} \left( \frac{(p' + p_\alpha)^2 + \gamma^2}{4p'p_\alpha} \right)^l \Gamma(1 + i\eta_\alpha) e^{i\alpha l}
\]

\[
\times \lim_{\gamma \to +0} \text{Im} \left\{ \frac{e^{-i\alpha l} (p' + p_\alpha + i\gamma)^{i\eta_\alpha - 1}}{(p' - p_\alpha + i\gamma)^{i\eta_\alpha + 1}} \right\}
\]

\[
\times _2 F_1 \left( -l, -l - i\eta_\alpha; 1 - i\eta_\alpha; \frac{(p' - p_\alpha)^2 + \gamma^2}{(p' + p_\alpha)^2 + \gamma^2} \right) + \gamma \left[ \ldots \right]
\]

\[
\psi_{p_\alpha l}^C(p') \text{ at low & high m} \text{c} \text{ switch: } \frac{4p'^2 p_\alpha^2}{(p'^2 + p_\alpha^2 + \gamma^2)^2} = \frac{(p' - p_\alpha)^2 + \gamma^2}{(p' + p_\alpha)^2 + \gamma^2}
\]

\[
\psi_{p_\alpha l}^C(p') = -2\pi e^{-\eta_\alpha \pi/2} (p' p_\alpha)^l \left[ \frac{\Gamma(l + 1 + i\eta_\alpha)\Gamma(\frac{1}{2})}{\Gamma(l + \frac{3}{2})} \right]
\]

\[
\times \lim_{\gamma \to +0} \left\{ \left[ \frac{2(p'^2 - (p_\alpha + i\gamma)^2)^{i\eta_\alpha}}{(p'^2 + p_\alpha^2 + \gamma^2)^l + i\eta_\alpha + 1} \right] \left( \frac{\eta_\alpha (p_\alpha + i\gamma)}{p'^2 - (p_\alpha + i\gamma)^2} - \frac{\gamma(l + i\eta_\alpha + 1)}{p'^2 + p_\alpha^2 + \gamma^2} \right) \right\}
\]

\[
\times _2 F_1 \left( \frac{l + i\eta_\alpha + 2}{2}, \frac{l + i\eta_\alpha + 1}{2}; l + \frac{3}{2}; \frac{4p'^2 p_\alpha^2}{(p'^2 + p_\alpha^2 + \gamma^2)^2} \right) + \gamma \left[ \ldots \right]
\]

\[
\text{Code will eventually be published}
\]