Towards Faddeev-AGS equations in a Coulomb basis in momentum space

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Deuteron-induced reactions are a point of the theoretical interest
To treat experimental results on the exotic nuclei beams with the deuterated targets in inverse kinematics.

Low energy
$E_{\text{lab}}$ from few up to 50 MeV/A.

Broad range of involved nuclei
From He isotopes up to Pb.

Main focus is on $(d,p)$
But the tools must analyze other three-body reaction channels (e.g. deuteron breakup) on the same footing.
Introduction & Motivation

$d + A$ reaction channels

Approximated three-body:
- elastic scattering,
- direct deuteron breakup $A(d, pn)A$ without core excitation,
- direct deuteron stripping $(d, p)$ or $(d, n)$ with nucleon capture without core excitation.

Approximated three-body with excitations:
- direct deuteron breakup and stripping with core excitations,
- non-elastic direct $(d, d')$ scattering with core excitations.

Many-body problems:
- core breakup,
- compound-nucleus reactions...
Deuteron-induced reactions as three-body problem


- Neglect the internal degrees of freedom of the nucleus to get 3-body problem.
- Use Faddeev-AGS approach to treat the problem.

To do later:
Take into account some target degrees of freedom:
- collective core excitations,
- more complicated internal dynamics.
Formal considerations about a 3-body system

3-body configurations & Jacobi coordinates

\[ A(pn) = d + A \]
\[ p(nA) \]
\[ n(Ap) \]

- Numerical indices: \( A \leftrightarrow 1, \ p \leftrightarrow 2, \ n \leftrightarrow 3. \)
- ‘Odd man out’ configuration notation: \( a(bc) \) labeled by \( a. \)

Denoting the two-body interactions

- \( V_a \equiv V_{bc}, \ i.e. \ a \ is \ a \ spectator. \)
3-body Hamiltonian

\[ H = H_0 + V_{np} + U_{nA} + v_{pA}, \quad H_0 = \frac{q_a}{2M_a} + \frac{p_a}{2\mu_a}. \]

Interactions between the particles

\[ v_{pA} = V_{pA}^C + v_{pA}^S, \quad V_{pA}^C = \frac{Z_A\alpha^2}{r}, \quad v_{pA}^S = U_{pA} + v_{pA}^{cd}, \]

- **V_{np}:** NN-interaction potential, e.g. chiral.
- **U_{nA}, U_{pA}:** phenomenological optical potentials, e.g. CH89.
- **v_{pA}^{cd}:** ‘short-range Coulomb interaction’, usually, the charged sphere potential.
Faddeev approach to a three-body problem

- $|\phi\rangle = \sum_{a=1}^{3} |\phi_a\rangle [= |\phi_1\rangle]$ is initial state $[A + (pn)]$.

Faddeev components $\psi_a$

$$|\Psi\rangle = |\phi\rangle + \sum_{a=1}^{3} g_0^C(E)V_a^S |\Psi\rangle, \quad g_0^C(E) = (E - H_0 - V_{pA}^C + i\varepsilon)^{-1},$$

$$|\Psi\rangle = \sum_a \delta_{a,1} |\phi_1\rangle + g_0^C(E)V_a^S |\Psi\rangle = \sum_a |\psi_a\rangle.$$

Faddeev-type equations

\[
\begin{cases}
|\psi_1\rangle = |\phi_1\rangle + g_0^C(E)V_1^S \sum_a |\psi_a\rangle \\
|\psi_2\rangle = g_0^C(E)V_2^S \sum_a |\psi_a\rangle \\
|\psi_3\rangle = g_0^C(E)V_3^S \sum_a |\psi_a\rangle
\end{cases} \quad \Rightarrow \quad \begin{cases}
|\psi_1\rangle = |\phi_1\rangle + g_0^C(E)t_1^S \sum_{a \neq 1} |\psi_a\rangle \\
|\psi_2\rangle = g_0^C(E)t_2^S \sum_{a \neq 2} |\psi_a\rangle \\
|\psi_3\rangle = g_0^C(E)t_3^S \sum_{a \neq 3} |\psi_a\rangle
\end{cases}
\]
Coulomb interaction in momentum space

Screening factor ($g_0^C$ becomes $g_0$, $\tilde{V}_C$ goes with the other potentials)

$$\tilde{V}_C = \frac{Z_1 Z_2 \alpha^2}{r} \exp(-\mu r^n).$$

- Various short-range amplitude corrections required.
Coulomb interaction in momentum space

Screening factor ($g_0^C$ becomes $g_0$, $\tilde{V}^C$ goes with the other potentials)

$$\tilde{V}^C = \frac{Z_1 Z_2 \alpha^2}{r} \exp(-\mu r^n).$$

Various short-range amplitude corrections required.

It works for $dp$ scattering

- $E_{p(lab)} = 9$ MeV.
- The most advanced screening techniques to the date.
- AV18 and AN18+UIX potentials.
A(d, p)B reaction with screened Coulomb interaction

It works for the nuclei up to $^{48}_{20}$Ca (Deltuva)


‘The charge wall’

- Discrepancy between the results of CDCC & Faddeev calculations becomes unreasonably large as the nucleus’ charge climbs up.
- Faddeev-AGS method hits the ‘charge wall’.
Faddeev approach on the way to the highly-charged nuclei

‘Total recall’

\[
\begin{align*}
\tilde{V}_C &= \frac{Z_1 Z_2 \alpha^2}{r} \exp(-\mu r^n) \\
g_0 &= \left[ E - H_0 + i\varepsilon \right]^{-1}
\end{align*}
\Rightarrow
\begin{align*}
V_C &= \frac{Z_1 Z_2 \alpha^2}{r} \\
g_C &= \left[ E - H_0 + i\varepsilon - V_C^{pA} \right]^{-1},
\end{align*}
\]

\( (H_0 = q_a/2M_a + p_a/2\mu_a) \).

In terms of the \( g_0 \) and \( g_C \) proper basis states (simplified notation):

\[
\begin{align*}
g_0 &= \frac{|\phi\rangle \langle \phi|}{E - H_0 + i\varepsilon}, \quad \Rightarrow \quad g_C &= \frac{|\psi_{k,\eta}^C\rangle \langle \psi_{k,\eta}^C|}{E - H_0 + i\varepsilon},
\end{align*}
\]

\[ \eta = Z_A e^2 \mu_p/k. \]
Coulomb function in momentum space

3D:

\[
\langle p \mid \psi^C_{k,\eta} \rangle = \psi^C_{k,\eta}(p) = -\frac{1}{2\pi^2} \lim_{\gamma \to +0} \frac{d}{d\gamma} \left\{ \frac{[p^2 + (\gamma - ik)^2]^{i\eta}}{[\gamma^2 + (p - k)^2]^{1+i\eta}} \right\}.
\]


In partial waves:

\[
\psi^C_{l,k,\eta}(p) = -\frac{2\pi e^{\pi\eta/2}}{pk} \lim_{\gamma \to +0} \frac{d}{d\gamma} \left\{ \left[ \frac{p^2 - (k + i\gamma)^2}{2pk} \right]^{i\eta} \frac{Q^\eta_l(\zeta)}{(\zeta^2 - 1)^{i\eta/2}} \right\},
\]

\[
\eta = Z_1 Z_2 \alpha^2 \mu / k; \quad \zeta = (k^2 + p^2 + \gamma^2) / 2kp.
\]

The two representations

The ‘regular’ representation \((p \neq k)\):

\[
\psi^C_{l,k,\eta}(p) = -\frac{4\pi \eta e^{-\pi \eta/2} q(pk)^l}{(p^2 + k^2)^{1+\eta}} \lim_{\gamma \to +0} \left[ p^2 - (k + i\gamma)^2 \right]^{-1+i\eta} \times \left[ \frac{\Gamma(1 + l + i\eta)}{(1/2)_{l+1}} \right] 2F_1 \left( \frac{2 + l + i\eta}{2}, \frac{1 + l + i\eta}{2}, l + 3/2; \frac{4k^2 p^2}{(p^2 + k^2)^2} \right).
\]

The ‘pole-proximity’ representation \((p \approx k)\):

\[
\psi^C_{l,k,\eta}(p) = \frac{2\pi}{p} \exp(-\pi \eta/2 + i\sigma_l) \left[ \frac{(p + k)^2}{4pk} \right]^l \lim_{\gamma \to +0} 2 \times \text{Im} \left[ \frac{\Gamma(1 + i\eta)e^{-i\sigma_l}(p + k)^{-1+i\eta}}{(p - k + i\gamma)^{1+i\eta}} \right] 2F_1 \left( -l, -l - i\eta, 1 - i\eta; \frac{(p - k)^2}{(p + k)^2} \right).
\]
Coulomb function in momentum space ($l = 0$ plot)

The two-body $t$-matrix elements in Coulomb basis

\[ g_0^C (E) t_{(l)a}^S = \frac{\langle \psi_{l,k',\eta'}^C | t_{(l)\alpha}^S | \psi_{l,k,\eta}^C \rangle \langle \psi_{l,k,\eta}^C | \psi_{l,k',\eta'}^C \rangle}{E - H_0 + i\varepsilon}, \]

\[ t_{a,l}^C (k', k, E) = \int dp' dp \psi_{l,k',\eta'}^C (p') \dagger t_{a,l}^S (p', p, E) \psi_{l,k,\eta}^C (p). \]

* Some indices omitted for the simplicity.

Pinch singularity in the elastic channel:

- $E = 2\mu k^2 = 2\mu k'^2$.
- Since $\gamma \to +0$, singularities of $\psi_{l,k',\eta'}^C$ and $\psi_{l,k,\eta}^C$ are pinching the integration contour.
If $t$ has a separable representation

$$
t_l^S(k', k, E) = \sum_{zy} u_{l,z}(k') \lambda_{l,zy}(E) u_{l,y}(k)\dagger,
$$

$$
t_l^C(k', k, E) = \sum_{zy} u_{l,z}^C(k') \lambda_{l,zy}(E) u_{l,y}^C(k)\dagger,
$$

$$
u_{l,z}^C(k') = \int \frac{dp'}{2\pi^2} u_{l,z}(p') \psi_{l,k',\eta'}(p')\dagger, \quad u_{l,y}^C(k)\dagger = \int \frac{dp}{2\pi^2} u_{l,y}(p)\dagger \psi_{l,k,\eta}(p).
$$

- Two independent integrals over $p$ and $p'$.
- Cauchy’s theorem.
- No pinch singularity!
Gel’fand-Shilov regularization for complex form-factors

\[ u^C_l(k) \propto \int_{a<0}^{b>0} \frac{f(y) \, dy}{y^{1+i\eta}} \equiv \int_a^b dy \, J_a(y), \quad (\text{e.g. } f(y) = y^2 + y + 1). \]

Subtract as many terms of Laurent expansion of \( f(y) \) around the integrand’s special point \( y = 0 \) as needed to split the integral and get the regular term, plus the analytically calculated terms.

Results: form-factors in Coulomb basis

\[ u_{l,1}^C(k) = \int \frac{dp}{2\pi^2} u_{l,1}(p) \psi_{l,k,\eta}(p)^*. \]

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Singularity contribution for $n + {}^{12}\text{C}$

Singularity contribution for $n + ^{208}\text{Pb}$

Faddeev-AGS equations in Coulomb basis (in progress)

Published abstract formalism

Faddeev-AGS equations in Coulomb basis for real two-body $t$-matrices and spinless particles.


To do:

- Take spin degrees of freedom into account.
- Deal with the 3-body singularity.
- Generalize the equations for the complex potentials.
- Take care about other implementation details, develop the algorithms and codes to solve the equations.

Objective:

- Calculate scattering observables and make the code available as open source.
Faddeev formalism is the theoretical tool to study deuteron-induced reactions. This formalism treats all possible three-body channels on the same footage. Momentum space is preferable due to the boundary conditions. Coulomb interaction can be treated properly by using the Coulomb basis in momentum space. Pinch singularity is avoided by choosing the two-body interactions in separable form. Mathematics and machinery are developed to compute Coulomb functions and matrix elements in Coulomb basis in momentum space. Work is in progress to cast the Faddeev-AGS equations in the Coulomb basis taking into account spin degrees of freedom and 3-body singularity in order to solve the equations numerically.